

§3. Equations with Compact Operators

Equations of the second kind

Let A be a compact operator defined on the normed space E into itself, the operator $T = I - A$ where I denotes the identity operator defines an operator equation called equation of the second kind, given as

$$T\varphi = (I - A)\varphi = \varphi - A\varphi = f,$$

where f is a given function of E and φ is the unknown function of E .

Theorem 1

the nullspace $N(T)$ of the operator T defined by

$$N(T) = \ker T = \{\varphi \in E; T\varphi = (I - A)\varphi = 0\},$$

is a closed finite dimensional subspace.

Proof

Indeed, it is known that the kernel $N(T)$ of a bounded operator T is a closed subspace of E , since, for all sequence φ_n in $N(T)$ converges to φ in E , we obtain φ in $N(T)$. Indeed, due to the boundedness of T we have

$$T\varphi_n = 0 \Rightarrow \lim_{n \rightarrow \infty} T\varphi_n = 0,$$

or still

$$T(\lim \varphi_n) = 0 \Rightarrow T(\varphi) = 0.$$

Hence, the nullspace $N(T)$ is closed.

On the other hand, all functions $\varphi \in N(T)$ must satisfy the equation

$$T\varphi = \varphi - A\varphi = 0, \text{ hence } A\varphi = \varphi.$$

Noting that, the restriction of the operator A to the subspace $N(T)$ coincides with the identity operator on $N(T)$. The operator A is compact from E to E and therefore also compact from $N(T)$ to $N(T)$, since $N(T)$ is closed. Evidently, for all bounded sequence φ_n in E in particular in $N(T)$ the sequence $A\varphi_n = \varphi_n$ contains a convergent subsequence $A\varphi_{n_k} = \varphi_{n_k}$ in the closed $N(T)$. Hence, the subspace $N(T)$ is finite dimensional.

Remark 1

The nullspace $N(T^n)$ of the operator T^n for all $n \in \mathbb{N}$, is a closed finite dimensional subspace.

Indeed, the operators T^n can be written in the form

$$T^n = (I - A)^n = I - A_n,$$

where A_n is a compact operator given by

$$A_n = \sum_{i=1}^n (-1)^{i+1} \binom{n}{i} A^i$$

Theorem 2

The sequence of nullspaces sets

$$N(T), N(T^2), \dots, N(T^n), \dots$$

is increasing and stationary sequence. In other words, the sequence contains uniquely a finite number of distinct sets, so there exists a nonnegative integer $p \in \mathbb{N}$, such that

$$\{0\} \subset N(T) \subset N(T^2) \subset \dots \subset N(T^p) = N(T^{p+1}) = \dots,$$

the number p is called the Riesz number of the compact operator A for the nullspaces sets $N(T^n)$.

Proof

Indeed, the inclusion is evident, since

$$\varphi \in N(T^n) \Rightarrow T^n \varphi = 0,$$

and therefore

$$T(T^n \varphi) = T^{n+1} \varphi = 0 \Rightarrow \varphi \in N(T^{n+1}).$$

Hence, the inclusion of sets

$$N(T^n) \subset N(T^{n+1}). \tag{1}$$

Suppose that there is no integer $p \in \mathbb{N}$, such that the sequence $N(T^n)$ is stationary, that is to say

$$N(T^m) \neq N(T^n), \text{ for all } m, n \in \mathbb{N}, \text{ with } m < n.$$

In other words, we have

$$\{0\} \subset N(T) \subset \dots \subset N(T^m) \subset N(T^{m+1}) \subset \dots \subset N(T^{n-1}) \subset N(T^n) \subset \dots$$

The relation $N(T^{n-1}) \subset N(T^n)$ between a closed subspaces involves the existence of an element φ_n in $N(T^n)$, with unit norm $\|\varphi_n\| = 1$, such that

$$\|\varphi_n - \varphi_{n-1}\| > \frac{1}{2}, \text{ for all } \varphi_{n-1} \in N(T^{n-1}).$$

Generally, for all sequence $\varphi_n \in N(T^n)$ and for all m, n such that $m < n$, we have the following relation

$$\begin{aligned} \|A\varphi_n - A\varphi_m\| &= \|\varphi_n - T\varphi_n - \varphi_m + T\varphi_m\| \\ &= \|\varphi_n - (\varphi_m - T\varphi_m + T\varphi_n)\| > \frac{1}{2}. \end{aligned} \quad (2)$$

Note that, the elements of the sequence $(\varphi_m - T\varphi_m + T\varphi_n)$ belong to the subspace $N(T^{n-1})$. Indeed, using the product by T^{n-1} we obtain

$$T^{n-1}(\varphi_m - T\varphi_m + T\varphi_n) = T^{n-1}\varphi_m - T^n\varphi_m + T^n\varphi_n.$$

Noting that

$$\varphi_m \in N(T^m) \subset N(T^{n-1}) \subset N(T^n) \quad \text{and} \quad \varphi_n \in N(T^n).$$

So

$$T^{n-1}\varphi_m = 0, \quad T^n\varphi_m = 0 \quad \text{and} \quad T^n\varphi_n = 0,$$

or still

$$T^{n-1}(\varphi_m - T\varphi_m + T\varphi_n) = 0.$$

The sequence φ_n is bounded, so by virtue of the compactness of the operator A , we can extract a convergent subsequence from the sequence $A\varphi_n$. Contradiction with the relation (2). Hence

$$N(T^{n-1}) = N(T^n).$$

It remains to demonstrate now the relation

$$N(T^n) = N(T^{n+1}).$$

Indeed, we get

$$\varphi \in N(T^{n+1}) \Rightarrow T^{n+1}\varphi = T^n(T\varphi) = 0,$$

or still

$$T\varphi \in N(T^n) = N(T^{n-1}),$$

that means

$$T\varphi \in N(T^{n-1}) \Rightarrow T^{n-1}(T\varphi) = T^n\varphi = 0 \Rightarrow \varphi \in N(T^n),$$

and therefore

$$N(T^{n+1}) \subset N(T^n).$$

Hence, there exists a nonnegative integer $p \in \mathbb{N}$, such that

$$\{0\} \subset N(T) \subset N(T^2) \subset \dots \subset N(T^p) = N(T^{p+1}) = N(T^{p+2}) = \dots,$$

where p is given by

$$p = \min\{k \in \mathbb{N}; \text{ such that } N(T^k) = N(T^{k+1})\}.$$

Theorem 3

The range space $R(T)$ of the operator T defined by

$$R(T) = \text{Im } T = T(E) = \{\psi; \exists \varphi \in E, T\varphi = \psi\},$$

is a closed subspace.

Proof

• It is known that, the range $R(T)$ of a linear operator T is a linear subspace.

Let f be an element of the closure $\overline{T(E)}$, then there exists a subsequence f_n of the set $T(E)$ such that

$$\lim_{n \rightarrow \infty} f_n = f.$$

In other words, There exists a sequence φ_n of E such that

$$T\varphi_n = f_n,$$

with the relation of convergence

$$\lim_{n \rightarrow \infty} T\varphi_n = \lim_{n \rightarrow \infty} f = f.$$

• Suppose that, the sequence φ_n is bounded, then there exists a subsequence $\varphi_{n(k)}$ such that, the sequence $A\varphi_{n(k)}$ converges. Hence, the convergence of the subsequence $\varphi_{n(k)}$ to an element φ in E . Say

$$\lim_{k \rightarrow \infty} \varphi_{n(k)} = \lim_{k \rightarrow \infty} T\varphi_{n(k)} - \lim_{k \rightarrow \infty} A\varphi_{n(k)} = \varphi \in E.$$

Due to the boundedness of the operator T , we get

$$\lim_{k \rightarrow \infty} T\varphi_{n(k)} = T(\lim_{k \rightarrow \infty} \varphi_{n(k)}) = T\varphi = f \in T(E) = \overline{T(E)}$$

• Suppose that, the sequence φ_n is not bounded, then we get

If $\varphi_n \in N(T)$, we have

$$T\varphi_n = 0 \Rightarrow \lim_{k \rightarrow \infty} T\varphi_n = 0 = f \in T(E) = \overline{T(E)},$$

as a linear subspace contains a null element.

If $\varphi_n \notin N(T)$, taking the subspace G of E spanned by φ_n and $N(T)$ defined as

$$G = \text{span} \{ \varphi_n + N(T) \}.$$

The subspace $N(T)$ is closed in G . Hence, there exists an element $\psi_n \in G$ with a unit norm $\|\psi_n\| = 1$ such that

$$\|\psi_n - \chi_n\| \geq \frac{1}{2}, \quad \forall \chi_n \in N(T),$$

with the following relation

$$\psi_n = a_n\varphi_n + \chi_n, \quad a_n \in \mathbb{R}, \quad \chi_n \in N(T).$$

Noting that, there is no subsequence $a_{n(k)}$ of the sequence a_n converges to the null element. For, if we suppose there exists a such subsequence, say $\lim_{k \rightarrow \infty} a_{n(k)} = 0$, we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} T\psi_{n(k)} &= \lim_{k \rightarrow \infty} (a_{n(k)}T\varphi_{n(k)}) + \lim_{k \rightarrow \infty} T\chi_{n(k)} \\ &= \lim_{k \rightarrow \infty} a_{n(k)} \cdot \lim_{k \rightarrow \infty} T\varphi_{n(k)} = 0 \cdot f = 0. \end{aligned}$$

In other words, there exists a subsequence $\psi_{n(j)}$ of the subsequence $\psi_{n(k)}$ of the bounded sequence ψ_n such that $A\psi_{n(j)}$ converges to an element ψ of E . This implies the convergence of the subsequence $\psi_{n(j)}$ to the same element ψ of E , for, we have

$$\lim_{j \rightarrow \infty} \psi_{n(j)} = \lim_{j \rightarrow \infty} T\psi_{n(j)} - \lim_{j \rightarrow \infty} A\psi_{n(j)} = \psi.$$

It is clear that, $T\psi = 0$. Hence $\psi \in N(T)$. Contradiction with the fact that

$$\|\psi_n - \chi_n\| \geq \frac{1}{2}, \quad \forall \chi_n \in N(T).$$

we can therefore conclude that a_n^{-1} is bounded. Say

$$a_n^{-1}\psi_n = \varphi_n + a_n^{-1}\chi_n,$$

Then, it comes

$$\begin{aligned} \lim_{n \rightarrow \infty} T(a_n^{-1}\psi_n) &= \lim_{n \rightarrow \infty} T\varphi_n + \lim_{n \rightarrow \infty} T(a_n^{-1}\chi_n) \\ &= \lim_{n \rightarrow \infty} T\varphi_n + 0 = f. \end{aligned}$$

The sequence $a_n^{-1}\psi_n$ is bounded as product of two bounded sequences a_n^{-1} and ψ_n . Hence there exists a subsequence $a_{n(k)}^{-1}\psi_{n(k)}$ such that $A(a_{n(k)}^{-1}\psi_{n(k)})$ converges to an element $a^{-1}\psi$ of E . This implies the convergence of the subsequence $a_{n(k)}^{-1}\psi_{n(k)}$ to the same element $a^{-1}\psi$ of E , for, we have

$$\lim_{k \rightarrow \infty} a_{n(k)}^{-1}\psi_{n(k)} = \lim_{k \rightarrow \infty} T(a_{n(k)}^{-1}\psi_{n(k)}) + \lim_{k \rightarrow \infty} A(a_{n(k)}^{-1}\psi_{n(k)}) = a^{-1}\psi \in E.$$

The operator T is continuous, then we write

$$\lim_{k \rightarrow \infty} T(a_{n(k)}^{-1}\psi_{n(k)}) = T(\lim_{k \rightarrow \infty} (a_{n(k)}^{-1}\psi_{n(k)})) = T(a^{-1}\psi) = f \in T(E) = \overline{T(E)}.$$

Hence, the result

$$T(E) = \overline{T(E)}.$$

Theorem 4

The sequence of range spaces sets

$$R(T), R(T^2), \dots, R(T^n), \dots$$

is decreasing and stationary sequence. In other words, the sequence contains uniquely a finite number of distinct sets, so there exists a nonnegative integer $q \in \mathbb{N}$, such that

$$\dots, \dots = R(T^{q+1}) = R(T^q) \subset \dots, \dots \subset R(T^2) \subset R(T) \subset X$$

The number q is called the Riesz number of the compact operator A for the range spaces sets $R(T^n)$.

Proof

Indeed, the inclusion is evident, since

$$\psi \in R(T^n) \Rightarrow \psi = T^n(\varphi) = T^{n-1}(T\varphi) = T^{n-1}\varphi_1 \in R(T^{n-1}),$$

and therefore

$$\psi = T^n\varphi \Rightarrow \psi = T^{n-1}\varphi_1.$$

Hence, the inclusion of sets

$$R(T^n) \subset R(T^{n-1}). \quad (1')$$

Suppose that there is no integer $q \in \mathbb{N}$, such that the sequence $R(T^n)$ is stationary, that is to say

$$R(T^m) \neq R(T^n), \text{ for all } m, n \in \mathbb{N}, \text{ with } m < n.$$

In other words, we have

$$\dots \subset R(T^n) \dots \subset R(T^{m+1}) \subset R(T^m) \subset \dots \subset R(T) \subset E$$

The relation $R(T^{m+1}) \subset R(T^m)$ between a closed subspaces involves the existence of an element ψ_m in $R(T^m)$, with unit norm $\|\psi_m\| = 1$, such that

$$\|\psi_m - \psi_{m+1}\| > \frac{1}{2}, \text{ for all } \psi_{m+1} \in R(T^{m+1}).$$

Generally, for all sequence $\psi_m \in R(T^m)$ and for all m, n such that $m < n$, we have the following relation

$$\begin{aligned} \|A\psi_m - A\psi_n\| &= \|\psi_m - T\psi_m - \psi_n + T\psi_n\| \\ &= \|\psi_m - (\psi_n - T\psi_n + T\psi_m)\| > \frac{1}{2}. \end{aligned} \quad (2')$$

Note that, the elements of the sequence $(\psi_n - T\psi_n + T\psi_m)$ belong to the subspace $R(T^{m+1})$. Indeed, we can write

$$\begin{aligned}\psi_n - T\psi_m + T\psi_n &= T^n\varphi - T(T^m\varphi) + T(T^n\varphi) \\ &= T^n\varphi - T^{m+1}\varphi + T^{n+1}\varphi.\end{aligned}$$

Hence

$$\psi_n, T\psi_n \in R(T^n) \subset R(T^{m+1}) \quad \text{and} \quad T\psi_m \in R(T^{m+1}).$$

The sequence ψ_m is bounded, so by virtue of the compactness of the operator A , we can extract a convergent subsequence from the sequence $A\psi_m$. Contradiction with the relation (2'). Hence

$$R(T^{m+1}) = R(T^m).$$

It remains to demonstrate now the relation

$$N(T^{m+2}) = N(T^{m+1}).$$

Indeed, the first inclusion $N(T^{m+2}) \subset N(T^{m+1})$ is always true following (1'), for the second one, we get

$$\psi \in R(T^{m+1}) \Rightarrow \psi = T^{m+1}\varphi = T(T^m\varphi) = T(T^{m+1}\varphi_1) = T^{m+2}\varphi_1 \in R(T^{m+2}),$$

or still

$$R(T^{m+1}) \subset R(T^{m+2}).$$

Hence, there exists a nonnegative integer $q \in \mathbb{N}$, such that

$$, \dots = R(T^{q+2}) = R(T^{q+1}) = R(T^q) \subset \dots R(T^2) \subset R(T) \subset X$$

where q is given by

$$q = \min\{k \in \mathbb{N}; \text{ such that } R(T^k) = R(T^{k+1})\}.$$

Lemma 1

The Riesz number p of the nullspaces sets $N(T^n)$ and the Riesz number q of the ranges spaces $R(T^n)$ are equal. Say

$$p = q$$

Proof

Suppose that, the Riesz numbers p and q are different, say $p \neq q$.

• **First case** $p < q$, we get

$$\{0\} \subset N(T) \subset \dots \subset N(T^p) = N(T^{p+1}) = \dots = N(T^{q-1}) = N(T^q) = \dots \quad (3)$$

and also

$$\dots = R(T^{q+1}) = R(T^q) \subset R(T^{q-1}) \subset \dots \subset R(T^p) \subset \dots \subset R(T) \subset X. \quad (4)$$

We can see that, there exists a function $\psi \in R(T^{q-1})$ such that $\psi \notin R(T^q)$, that is to say

$$\psi = T^{q-1}\varphi \in R(T^{q-1}),$$

the composition by the operator T of both sides, gives us

$$T\psi = T^q\varphi \in R(T^q) = R(T^{q+1}),$$

this relation shows that, there exists a function φ_1 such that

$$T\psi = T^q\varphi = T^{q+1}\varphi_1,$$

or still

$$T^{q+1}\varphi_1 - T^q\varphi = 0.$$

Hence, we obtain

$$T^q(T\varphi_1 - \varphi) = 0 \Rightarrow T(\varphi_1) - \varphi \in N(T^q) = N(T^{q-1}). \quad (5)$$

It is to remark that the relation (5) gives us

$$T(\varphi_1) - \varphi \in N(T^{q-1}) \Rightarrow T^{q-1}(T\varphi_1 - \varphi) = 0 \Leftrightarrow T^q\varphi_1 = T^{q-1}\varphi = \psi,$$

this implies that $\psi = T^q\varphi_1 \in R(T^q)$, contradiction with the fact that $\psi \notin R(T^q)$.

• **Second case** $q < p$, we get

$$\{0\} \subset N(T) \subset \dots \subset N(T^q) \subset \dots \subset N(T^{p-1}) \subset N(T^p) = N(T^{p+1}) = \dots \quad (3')$$

and also

$$\dots = R(T^p) = R(T^{p-1}) = \dots = R(T^q) \subset R(T^{q-1}) \subset \dots \subset R(T) \subset X. \quad (4')$$

We can see that, there exists a function $\varphi \in N(T^p)$ such that $\varphi \notin N(T^{p-1})$, that is to say

$$T^{p-1}\varphi \in R(T^{p-1}) = R(T^p) = R(T^q),$$

this relation shows that, there exists functions φ_1 and φ_2 such that

$$T^{p-1}\varphi = T^p\varphi_1 = T^q\varphi_2, \quad (5')$$

the composition by the operator T of both sides and the relation $\varphi \in N(T^p)$ give us

$$T^p\varphi = T^{p+1}\varphi_1 = T^{q+1}\varphi_2 = 0,$$

this implies that

$$\varphi_1 \in N(T^{p+1}) = N(T^p).$$

Hence, it comes

$$T^{p+1}\varphi_1 = T^p\varphi_1 = 0.$$

It is to remark that the relation (5') gives us $T^p\varphi_1 = T^{p-1}\varphi = 0$, this implies that $\varphi \in N(T^{p-1})$, contradiction with the fact that $\varphi \notin N(T^{p-1})$.

Theorem 5

The subspaces $N(T^r)$ and $R(T^r)$ are supplementary. That is to say

$$X = \ker T^r \oplus \text{Im } T^r = N(T^r) \oplus R(T^r),$$

where $r = p = q$ is the Riesz number.

Proof

- For all element $\psi \in X$, we get

$$\psi \in X \Rightarrow T^r\psi \in R(T^r) = \dots = R(T^{2r}).$$

This relation implies the existence of a function φ , such that

$$T^r\psi = T^{2r}\varphi \Rightarrow T^r(\psi - T^r\varphi) = 0,$$

or still,

$$(\psi - T^r \varphi) = \chi \in N(T^r).$$

Therefore, we have

$$\psi = \chi + T^r \varphi, \text{ with } \chi \in N(T^r) \text{ and } T^r \varphi \in R(T^r).$$

- For all element $\psi \in N(T^r) \cap R(T^r)$, we get

$$\psi \in R(T^r) \text{ and } \psi \in N(T^r),$$

This relation implies the existence of a function φ , such that

$$\psi = T^r \varphi \text{ and } T^r \psi = T^{2r} \varphi = 0,$$

or still,

$$\varphi \in N(T^{2r}) = \dots, \dots = N(T^r).$$

Therefore, we have

$$\psi = T^r \varphi = 0.$$

Lemma 2

The operator $T = I - A$ is injective if and only if, the operator T^t is injective for all $r \in \mathbb{N}$.

Proof

- Supposing that, The operator T is injective then, for all $r \in \mathbb{N}$, we have

$$\begin{aligned} T^r \varphi_1 = T^r \varphi_2 &\Rightarrow T(T^{r-1} \varphi_1) = T(T^{r-1} \varphi_2) \Rightarrow T^{r-1} \varphi_1 = T^{r-1} \varphi_2 \\ &\Rightarrow T(T^{r-2} \varphi_1) = T(T^{r-2} \varphi_2) \Rightarrow T^{r-2} \varphi_1 = T^{r-2} \varphi_2 \\ &\Rightarrow \dots T(T \varphi_1) = T(T \varphi_2) \Rightarrow T \varphi_1 = T \varphi_2 \\ &\Rightarrow \varphi_1 = \varphi_2. \end{aligned}$$

Hence, the operator T^r is injective.

- Supposing that, The operator T^r is injective for all $r \in \mathbb{N}$, then we have

$$\begin{aligned} T \varphi_1 = T \varphi_2 &\Rightarrow T^{r-1}(T \varphi_1) = T^{r-1}(T \varphi_2) \Rightarrow T^r \varphi_1 = T^r \varphi_2 \\ &\Rightarrow \varphi_1 = \varphi_2. \end{aligned}$$

Hence, the operator T is injective. That is to say

$$\{0\} = N(T) = N(T^2) = \dots = N(T^r) = \dots, \dots$$

Lemma 3

The operator $T = I - A$ is surjective if and only if, the operator T^t is surjective for all $r \in \mathbb{N}$.

Proof

- Suppose that, the operator T is surjective then, for all $r \in \mathbb{N}$, we have

$$\begin{aligned} \forall \psi \in X, \exists \varphi_1 \in X; \psi = T\varphi_1 &\Rightarrow \exists \varphi_2 \in X; \varphi_1 = T\varphi_2 \\ &\Rightarrow \psi = T\varphi_1 = T(T\varphi_2) = T^2\varphi_2 \\ &\Rightarrow \dots \exists \varphi_r \in X; \varphi_{r-1} = T\varphi_r \\ \psi = T\varphi_1 = T(T\varphi_2) = \dots = T(T^{r-1}\varphi_r) &= T^r\varphi_r \\ &\Rightarrow \forall \psi \in X, \exists \varphi_r \in X; \psi = T^r\varphi_r \end{aligned}$$

Hence, the operator T^r is surjective.

- Suppose that, the operator T^r is surjective for all $r \in \mathbb{N}$, then we have

$$\begin{aligned} \forall \psi \in X, \exists \varphi_1 \in X; \psi = T^r\varphi_1 &\Rightarrow T^r\varphi_1 = T(T^{r-1}\varphi_1) = T\varphi \\ &\Rightarrow \forall \psi \in X, \exists \varphi = T^{r-1}\varphi_1 \in X; \psi = T\varphi. \end{aligned}$$

Hence, the operator T is surjective. That is to say

$$X = R(T) = R(T^2) = \dots = R(T^r) = \dots, \dots$$

Theorem 6

Let A be a compact operator defined from the normed space X into itself then, the operator $T = I - A$ is an injective operator if and only if, $T = I - A$ is surjective. Besides the inverse operator $T^{-1} = (I - A)^{-1}$ defined from X into X is bounded.

Proof

It is known that, for all Riesz number $r = p = q$, The subspaces $N(T^r)$ and $R(T^r)$ are supplementary. Say

$$X = N(T^r) \oplus R(T^r).$$

- The injection of the operator T implies the one of T^r . Hence the surjection of the operator T^t which it assures us the surjection of the operator T .
- The surjection of the operator T implies the one of T^r . Hence the injection of the operator T^t which it assures us the injection of the operator T .
- The injection of the operator T or the surjection of the operator T implies the bijection of the operator $T = (I - A)$. This operator is bounded then it has a bounded inverse $T^{-1} = (I - A)^{-1}$.

Theorem 7

Let A be a compact operator from a normed space E into itself then, the nonhomgene equation

$$T\varphi = \varphi - A\varphi = f \tag{6}$$

admits a unique solution $\varphi \in X$, for all $f \in X$, if and only if the homogene equation

$$T\varphi = \varphi - A\varphi = 0 \tag{6'}$$

admits uniquely a trivial solution $\varphi = 0$.

Proof

Indeed, suppose that the equation (6) admits a solution for all $f \in X$, it wants to say that the operator T is surjective and the Riesz number r is null. Hence the operator T is injective. In other words, the equation (6') admits the trivial solution $\varphi = 0$.

Reciprocally, suppose that the equation (6') admits uniquely the trivial solution $\varphi = 0$, it wants to say that the operator T is injective and the Riesz number r is null. Hence, the operator T is surjective and therefore this operator is bijective. In other words, the existence and the uniqueness of the solution of the equation (6).

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