§1. Compactness

Covers

A class of subsets $H = \{U_j\}$ of E is called a cover of a set G of E if we have

$$G \subset \bigcup_j U_j$$

Example 1

Let H be a class of subsets given by $H = \{[m, m+1[, \ m \in \mathbb{Z}\}$. Then H is a cover of the real line $\mathbb{R}$ because for every point in $\mathbb{R}$ there exists an interval of H contains this point.

Example 2

Let H be a class of subsets given by $H = \{]m, m+1[, \ m \in \mathbb{Z}\}$. Then H is not a cover of the real line $\mathbb{R}$ because the points m in $\mathbb{R}$ do not belong to any interval of H.

Open covers

The cover $H = \{U_j\}$ of G is called an open cover if each U_j is open.

Example 3

Let H be a class of open subsets given by $H = \{]m, m+2[, \ m \in \mathbb{Z}\}$. Then H is an open cover of the real line $\mathbb{R}$ because for every point in $\mathbb{R}$ there exists an open interval of H contains this point.

Finite covers

The cover $H = \{U_{j_1}, U_{j_2}, ..., U_{j_n}\}$ of G is called a finite cover of G if

$$G \subset U_{j_1} \cup U_{j_2} \cup ... \cup U_{j_n}.$$

Theorem 1 (**Heine-Borel**)

Every open cover $H = \{U_j\}$ of a closed and bounded interval $G = [a, b]$ has a finite cover. That is to say $G \subset \bigcup_{j \in J} U_j$ involves that there exists a finite number of an open sets $U_{j_1} \cup U_{j_2} \cup ... \cup U_{j_n}$ such that

$$G \subset U_{j_1} \cup U_{j_2} \cup ... \cup U_{j_n}.$$

Compact set

A subset G of a topological space E is called compact if every open cover $H = \{U_j\}$ of G contains a finite subcover of G. In other words, for every family $\{U_j\}$, $j \in J$, of an open sets with the property

$$G \subset \bigcup_{j\in J} U_j$$

there exists a finite subfamily $\{U_{j(k)}\}$, $j(k) \in J$, $k = 1, 2, ..., n$, such that

$$G \subset \bigcup_{k=1}^{n} U_{j(k)}.$$

or still

$$\exists\ U_{j_1}, U_{j_2}, ..., U_{j_n} \in H \quad \text{such that } G \subset U_{j_1} \cup U_{j_2} \cup ... \cup U_{j_n}.$$

Remark 1

The idea of the compactness is motived by the Heine-Borel theorem of a closed and bounded interval.

Theorem 2

Every closed and bounded real interval $G = [a, b]$ is compact.

Remark 2

The open bounded interval $]a, b[$, the open closed bounded interval $]a, b]$, and the closed open bounded interval $[a, b[$ are not compact.

Remark 3

Every finite subset $G = \{\varphi_1, \varphi_2, .., \varphi_n\}$ of a topological space E is compact

Theorem 3

A closed subset V of a compact set G is also compact

Proof

Let $H = \{U_j\}$ be an open cover of the closed subset V, that is to say $V \subset \cup U_j$, from the relation $G = V \cup V^c$ we get an open cover of the compact set G, say $G \subset \cup U_j \cup V^c$. Hence there exists a finite cover of G

$$G \subset U_{j_1} \cup U_{j_2} \cup ... \cup U_{j_n} \cup V^c.$$

Due to the disjointedly of the sets V and V^c, the subfamily $\{U_{j_1}, U_{j_2}, ..., U_{j_n}\}$ represents a finite cover of the subset V. In other words

$V \subset U_{j_1} \cup U_{j_2} \cup ... \cup U_{j_n}$. Hence V is compact.

Remark 4

The subset V must be closed, in order to obtain an open subset V^c as an element of the covers of G.

Sequentially compact set

A subset G of a topological space E is called sequentially compact if and only if, every sequence of elements of G contains a subsequence which converges to an element in G.

Remark 5

There is no relation between the compactness and the sequential compactness in the topological space. Because, the adherent point of a set G in the topological space is not necessary a limit of convergent subsequence.

Remark 6

An adherent point of a set G in the normed space is a limit of convergent subsequence.

Theorem 4

Let G be a subset of a normed space E.Then the following statements are equivalent

(i) G is compact

(ii) G is sequentially compact.

Remark 7

Every bounded sequence (x_n) in the closed bounded interval $G = [a, b]$ has a convergent subsequence (x_{n_k}) in G.

Totally bounded set

A subset G of a normed space E is called a totally bounded if for each real positive $\varepsilon > 0$, there exists a finite number of elements $\varphi_1, \varphi_2, ...\varphi_n$ in G such that

$$G \subset \bigcup_{j=1}^{n} B(\varphi_j, \varepsilon).$$

Net for set

A finite set of points $\{\varphi_1, \varphi_2, ...\varphi_n\}$ of a normed space E is called $\varepsilon - net$ for a subset G of E, if for every point ψ in G there exists a φ_j such that

$$\| \psi - \varphi_j \| < \varepsilon$$

Remark 7

Each element ψ from a totally bounded set G has a distance less than ε from least one of the elements $\varphi_1, \varphi_2, ..., \varphi_n$. In other words, the totally bounded set G has a $\varepsilon - net$ for all $\varepsilon > 0$.

Lemma 1

Each sequentially compact set G is totally bounded

Proof

Indeed, suppose that G is not totally bounded then there exist a real positive $\varepsilon > 0$ and a point $\psi_1 \in G$ with the property

$$\| \psi_1 - \varphi_j \| \geq \varepsilon \text{ for all } j = 1, 2, .., n.$$

Then there exists $\psi_2 \in G$ such that

$$\| \psi_1 - \psi_2 \| \geq \varepsilon \text{ and } \| \psi_2 - \varphi_j \| \geq \varepsilon \text{ for all } j = 1, 2, .., n,$$

for otherwise $\{\varphi_1, \varphi_2, .., \varphi_n, \psi_1\}$ would be $\varepsilon - net$ for G. Continuing in this manner, we arrive at a sequence $\{\psi_1, \psi_2, .., \psi_n, ..\}$ with the property that

$$\| \psi_i - \varphi_j \| \geq \varepsilon \text{ for all } i \neq j.$$

Thus, the sequence (ψ_n) does not contain a convergent subsequence. In other words G is not sequentially compact.

Remark 8

The compact sets G of a normed space E are bounded, closed and complete.

Relatively compact set

A subset G of a normed space E is called relatively compact if its closure $\overline{G}$ is compact.

Remark 9

A subset G of a normed space E is relatively compact if and only if each sequence φ_n of elements from G contains a convergent subsequence. Further, relatively compact sets are totally bounded.

Theorem 5

A bounded and finite dimensional subset G of a normed space E is relatively compact

Proof This follows from the Belzano-Weirstrass theorem.

Compactness in $C(K)$

Theorem 6 (Arzela-Ascoli)

A set $G \subset C(K)$ is relatively compact if and only if it is bounded and equicontinuous. Say, If there exists a constant M such that

$$| \varphi(x) | \leq M \quad \text{for all } x \in K \text{ and all } \varphi \in G.$$

Further, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $\varphi \in G$, we have

$$| \varphi(x) - \varphi(y) | < \varepsilon \text{ for all } x, y \in K \text{ with } | x - y | < \delta.$$

The set $C(K)$ designates the space of continuous real or complex valued functions defined in a compact set $K \subset \mathbb{R}^n$, furnished with the maximum norm

$$\| \varphi \|_\infty = \max_{x \in K} | \varphi(x) | \,.$$

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