§2. Compact Operators

Compact linear operators

A linear operator A defined from a normed space E into a normed space F is called a linear compact operator or completely continuous linear operator if for every bounded subset G of E, the image $A(G)$ is relatively compact in F. In other words, the closure $\overline{A(G)}$ is compact.

Theorem 1 (Compactness criterion)

A linear operator A defined from a normed space E into a normed space F is called a linear compact operator or completely continuous linear operator if and only if for every bounded sequence φ_n in E, the sequence $A\varphi_n$ in F has a convergent subsequence.

Proof

Let φ_n be a bounded sequence in E, since the operator A is compact, then the set $\{A\varphi_n\}$ is relatively compact in F where this property shows that $A\varphi_n$ contains a convergent subsequence.

Conversely, let us consider any bounded subset G in E and let ψ_n be any sequence in $A(G)$. Then there exists a bounded sequence φ_n in G, such that

$$\psi_n = A\varphi_n.$$

By assumption, $A\varphi_n = \psi_n$ contains a convergent subsequence ψ_{n_k} in F. Thus $A(G)$ is relatively compact, because for any bounded sequence ψ_n in $A(G)$ there exits a convergent subsequence ψ_{n_k} in F. In other words, for all bounded set $G \subset E$, the set $A(G)$ is relatively compact in F. Hence A is compact.

Theorem 2

The linear combination $A = \alpha A_1 + \beta A_2$ of compact operators A_1 and A_2 is a compact operator, for every scalars α and β.

Proof

Let φ_n be a bounded sequence in E and let $A\varphi_n$ be a sequence in F, then

$$A\varphi_n(x) = \alpha A_1\varphi_n(x) + \beta A_2\varphi_n(x), \text{ with } \varphi_n \in E, \; n \in \mathbb{N}.$$

The operators A_1 and A_2 are compact, one can extract from $A_1\varphi_n$ and $A_2\varphi_n$ two convergent subsequences which give by their sum a convergent subsequence of $A\varphi_n$. Hence A is compact.

Theorem 3

The product AB of two bounded operators A and B is compact if either of operators A or B is compact.

Proof

Let φ_n be a bounded sequence in E, then if we consider B as a bounded operator the sequence $B\varphi_n(x)$ is bounded, and from the compactness of the operator A gives a convergent subsequence $A(B\varphi_{n_k}(x))$ of $A(B\varphi_n(x))$. Hence the operator AB is compact.

On the other hand, if we consider B as a compact, one can extract from $B\varphi_n(x)$ a convergent subsequence $B\varphi_{n(k)}(x)$, and from the boundedness of the operator A gives the convergence of the sequence $A(B\varphi_{n(k)}(x))$. Hence the operator AB is compact.

Theorem 4

A sequence A_n of compact operators defined from a normed space E into a Banach space F converges uniformly to an operator A, say,

$$\lim_{n\to\infty} \|A_n - A\| = 0.$$

Then the limit operator A is compact.

Proof

Let φ_n be a bounded sequence in E, the operator $A_1$is compact, then one can extract from the sequence $A_1\varphi_n$ a convergent subsequence, say φ_n^1 a subsequence from φ_n such that $A_1\varphi_n^1$ converges.

In the same way, we can extract from the sequence $A_2\varphi_n^1$ a convergent subsequence , say φ_n^2 a subsequence from φ_n^1 such that $A_2\varphi_n^2$ converges.

Noting that, we obtain from the bounded sequence φ_n a subsequence φ_n^2 such that $A_1\varphi_n^2$ and $A_2\varphi_n^2$ both converge.

Continuing in this way, we see that, for the compact operators $A_1, A_2, .., A_p$, there exists a nested subsequences

$$\varphi_n^p \subset ....\varphi_n^2 \subset \varphi_n^1 \subset \varphi_n,$$

such that, the sequences $A_k\varphi_n^p$ converge for all $k = 1, 2, ..., p$.

In order to show the compactness of the operator limit A, we must use the completeness of the space F and showing that the sequence $A\varphi_n^p$ is Cauchy sequence.

Noting that the sequence φ_n is bounded, say $\|\varphi_n\| \leq M$ for all n. Hence $\| \varphi_n^p \| \leq M$ for each n and p. Choose $n = p$ so that

$$\|A_n - A\| < \frac{\varepsilon}{3M}.$$

Since the sequence $A_n\varphi_n^p$ is Cauchy, because it converges there is N such that, for all $p > N$ and $q > N$, we get

$$\|A_n\varphi_n^p - A_n\varphi_n^q\| < \frac{\varepsilon}{3}.$$

Hence we obtain

$$\begin{aligned}
\| A\varphi_n^p - A\varphi_n^q \| &= \|A\varphi_n^p - A_n\varphi_n^p + A_n\varphi_n^p - A_n\varphi_n^q + A_n\varphi_n^q - A\varphi_n^q\| \\
&\leq \|A\varphi_n^p - A_n\varphi_n^p\| + \|A_n\varphi_n^p - A_n\varphi_n^q\| + \|A_n\varphi_n^q - A\varphi_n^q\| \\
&\leq \|A_n - A\| \, \|\varphi_n^p\| + \|A_n\varphi_n^p - A_n\varphi_n^q\| + \|A_n - A\| \, \|\varphi_n^q\| \\
&\leq \frac{\varepsilon}{3M}M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M}M = \varepsilon.
\end{aligned}$$

Remembering that, due to the completeness of the space F, the Cauchy sequence $A\varphi_n^p$ converges as a subsequence of $A\varphi_n$ where φ_n^p is a subsequence of an arbitrary bounded sequence φ_n. Hence the compactness of the operator A.

Theorem 5 *(finite dimensional range)*

Let A be a bounded operator defined from E into F with the range $A(E)$ has a finite dimension $\dim A(E) < \infty$, *then the operator A is compact.*

Proof

Indeed, for all bounded set G in E, the range $A(G)$ is a bounded set in the finite dimensional space $A(E)$. Hence $A(G)$ is relatively compact, it follows that A is a compact operator.

Theorem 6 *(finite dimensional domain)*

Let A be a bounded operator defined from E into F with the domain E has a finite dimension $\dim E < \infty$, *then the operator A is compact.*

Proof

Indeed, the space E has a finite dimension $\dim E < \infty$ implies the finite dimensional range $A(E)$, say

$$\dim A(E) \leq \dim E,$$

it follows that A is a compact operator.

Lemma 1

Let G be a closed subspace in the normed space E such that, $G \neq E$ then there exists an element $\varphi \in E$ with $\|\varphi\| = 1$ such that, for all $\psi \in G$, we have

$$\|\varphi - \psi\| \geq \alpha, \text{ with } 0 < \alpha < 1$$

Proof

Indeed, let f be an element of E such that $f \notin G$ then, we get

$$\inf_{h \in G} \|f - h\| = \beta > 0,$$

choosing an element g belongs to G such that,

$$\beta \leq \|f - g\| \leq \frac{\beta}{\alpha}.$$

Define the vector φ by

$$\varphi = \frac{f - g}{\|f - g\|},$$

this vector φ has a unit norm $\|\varphi\| = 1$, besides, for all $\psi \in G$ we get

$$\begin{aligned} \|\varphi - \psi\| &= \left\| \frac{f - g}{\|f - g\|} - \psi \right\| \\ &= \frac{1}{\|f - g\|} \|f - [g + (\|f - g\| \psi)]\| \\ &\geq \frac{\beta}{\|f - g\|} \geq \alpha. \end{aligned}$$

Theorem 7

The identity operator I defined from a normed space E into E is compact if and only if the space E has a finite dimension.

Proof

let $\varphi_1$be an element of E, such that $\|\varphi_1\| = 1$, then the set of finite dimension $G_1 = span\{\varphi_1\}$ represents a closed subspace of E. So there exists an element $\varphi_2 \in E$, such that $\|\varphi_2\| = 1$ and $\|\varphi_1 - \varphi_2\| \geq \frac{1}{2}$. By the same way we take a closed subspace $G_2 = span\{\varphi_1, \varphi_2\}$ and finding an element $\varphi_3 \in E$ such that $\|\varphi_2\| = 1$ with $\|\varphi_1 - \varphi_3\| \geq \frac{1}{2}$ and $\|\varphi_2 - \varphi_3\| \geq \frac{1}{2}$. One repeat the same procedure until the obtaining of a sequence φ_n verifying $\|\varphi_n\| = 1$ and $\|\varphi_m - \varphi_n\| > \frac{1}{2}$, for all $m \neq n$.

Noting that, the sequence φ_n is bounded but does not contain any convergent subsequence. Hence the operator $I\varphi_n = \varphi_n$ is not compact.

Corollary 1

The closed unit ball $B(0,1)$ in the normed space E of infinitely dimensional is not compact.

Indeed, $B(0,1)$ is bounded but cannot be compact; thus

$$I(B(0,1) = B(0,1) = \overline{B(0,1)},$$

is not relatively compact.

Corollary 2

A bounded operator A in a normed space E is not generally a compact operator.

Indeed, see the Identity operator $A = I$ in the infinitely dimensional normed space E.

Theorem 8

The integral operator A defined from $C(G)$ into $C(G)$

$$A\varphi(x) = \int_G k(x,y)\varphi(y)dy, \quad x, y \in G$$

with continuous kernel $k(x,y)$ is a compact operator.

Proof

Let E be a bounded set of $C(G)$ then, for each $\varphi \in E$, we have

$$\|\varphi\| \leq M,$$

besides, for all $x \in G$ and $\varphi \in E$, we get

$$\begin{aligned} |A\varphi(x)| &= \left|\int_G k(x,y)\varphi(y)dy\right| \\ &\leq M\,|G| \max_{x,y\in G} |k(x,y)|\,. \end{aligned}$$

It follows that $A(E)$ is bounded.

By assumption, the kernel $k(x,y)$ is continuous over the compact $G \times G$, thus it is uniformly continuous and therefore

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x,y,z \in G,\ |x-y| < \delta \Rightarrow |k(x,z) - k(y,z)| < \frac{\varepsilon}{M\,|G|}.$$

Hence, for each $\varphi \in E$ and $x, y \in G$, with $|x-y| < \delta$

$$\begin{aligned} |A\varphi(x) - A\varphi(y)| &= \left|\int_G (k(x,z) - k(y,z))\varphi(z)dz\right| \\ &< \frac{\varepsilon}{M\,|G|} M\,|G| = \varepsilon \end{aligned}$$

This relation expresses that $A(E)$ is equicontinuous. Hence $A(E)$ is relatively compact, so by Arzela-Ascoli's theorem A is compact.

Weakly singular kernel

The kernel $k(x,y)$ is said to be weakly singular if it is defined continuous on $G \times G \subset \mathbb{R}^n \times \mathbb{R}^n$ for all $x \neq y$ and there exist a positive constants M and $\alpha \in]0,n]$ such that

$$|k(x,y)| < \frac{M}{|x-y|^{n-\alpha}}, \quad x,y \in G, \quad x \neq y.$$

In other words,

$$\forall x,y \in G,\ x \neq y,\ \exists M > 0,\ |k(x,y)| < \frac{M}{|x-y|^{n-\alpha}}, \quad 0 < \alpha \leq n$$

Theorem 9

The integral operator A defined from $C(G)$ into $C(G)$ with weakly continuous kernel is a compact operator.

proof
Noting that, the integral operator

$$A\varphi(x) = \int_G k(x,y)\varphi(y)dy, \quad x, y \in G$$

exists as an improper integral, due to the weakly continuous kernel

$$|k(x,y)\varphi(y)| \leq M \|\varphi\| |x-y|^{n-\alpha},$$

further,

$$\int_G |x-y|^{n-\alpha} dy \leq \omega_n \int_0^d \rho^{\alpha-n}\rho^{n-1} d\rho = \frac{\omega_n}{\alpha} d^\alpha,$$

where ω_n designates the surface area of the unit sphere in $\mathbb{R}^n$ and d the diameter of the set G.

Let us construct a sequence of compact operators A_p which converges to the integral operator A, such that

$$\lim_{n\to\infty} \|A_p - A\| = 0.$$

choosing now a linear continuous function h defined on $[0,\infty[$ into $\mathbb{R}$, by

$$h(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ 2t-1 & \text{if} \quad \frac{1}{2} \leq t \leq 1 \\ 1 & \text{if} \quad 1 \leq t < \infty \end{cases},$$

The function $k_p(x,y)$ defined on $G \times G$ into $\mathbb{R}$, by

$$k_p(x,y) = \begin{cases} h(p|x-y|)k(x,y) & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

is a continuous kernel for each $p \in \mathbb{N}$. Hence the integral operators A_p such that

$$A_p\varphi(x) = \int_G k_p(x,y)\varphi(y)dy, \quad x, y \in G,$$

are compact.

Besides, for all $x \in G$, we get

$$\begin{aligned} |A_p\varphi(x) - A\varphi(x)| &= \left| \int_G [k_p(x,y) - k(x,y)]\varphi(y)dy \right| \\ &= \left| \int_{G\cap|x-y|<\frac{1}{p}} \{h(p\,|x-y|) - 1\} k(x,y)\varphi(y)dy \right| \\ &\leq M\,\|\varphi\|\,\omega_n \int_0^{\frac{1}{p}} \rho^{\alpha-n}\rho^{n-1}d\rho \\ &\leq M\,\|\varphi\|\,\frac{\omega_n}{\alpha p^\alpha}. \end{aligned}$$

It is simple to see that the convergence $A_p\varphi$ to $A\varphi$ is uniform, so it follows that,

$$\|A - A_p\| \leq M\frac{\omega_n}{\alpha p^\alpha} \to 0, \text{ when } p \to \infty,$$

and thus A is compact operator.

Theorem 10

The integral operator A defined from the normed space $C(\partial G)$ into $C(\partial G)$ with continuous or weakly continuous kernel is a compact operator, where under ∂G we designate a regular boundary of the set G.

Bibliography

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