§4. Fredholm's Alternative

Theorem 1 *(Riesz Representation)*

Let φ *be a bounded linear form defined from a Hilbert space* H *into* $K = (\mathbb{R}$ *or* $\mathbb{C})$ *then, there exists a unique element* y *in* H *such that, for all* x *in* H, *we have*

$$\varphi(x) = \langle x, y \rangle .$$

In other words, all linear form in a Hilbert space H *is a inner product in* H. *Say*

$$\forall x \in H \quad \exists! y \in H, \qquad \varphi(x) = \langle x, y \rangle .$$

Proof

1. *Existence*

Suppose that, for all $x \in H$, $\varphi(x) = 0$, then the space H coincide with the null space $N(\varphi)$, so we may choose $y = 0$. That is to say

$$\varphi(x) = 0 = \langle x, 0 \rangle , \quad \text{for all } x \in H.$$

Suppose that, $\varphi(x) \neq 0$ then the null space $N(\varphi)$ is a closed proper subspace of H. Say $N(\varphi) \subset H$, so we may find a non zero element z in the orthogonal subspace $N^{\perp}(\varphi)$ of the null space $N(\varphi)$. Say

$$H = N(\varphi) \oplus N^{\perp}(\varphi).$$

For all $x \in H$ and $z \in N^{\perp}(\varphi)$ we get

$$x - \frac{\varphi(x)}{\varphi(z)} z \in N(\varphi),$$

namely

$$\varphi\left(x - \frac{\varphi(x)}{\varphi(z)} z\right) = \varphi(x) - \frac{\varphi(x)}{\varphi(z)} \varphi(z) = 0.$$

Therefore, we write

$$\left\langle x - \frac{\varphi(x)}{\varphi(z)} z, z \right\rangle = 0,$$

or still

$$\langle x, z \rangle = \frac{\varphi(x)}{\varphi(z)} \|z\|^2 ,$$

hence, it comes

$$\begin{aligned}\varphi(x) &= \frac{\varphi(z)}{\|z\|^2}\langle x, z\rangle \\ &= \left\langle x, \frac{\overline{\varphi(z)}}{\|z\|^2} z \right\rangle.\end{aligned}$$

In other words, we obtain

$$y = \frac{\overline{\varphi(z)}}{\|z\|^2} z.$$

2. *Uniqueness*

Suppose that y_1 and y_2 be two elements of H such that

$$\varphi(x) = \langle x, y_1\rangle = \langle x, y_2\rangle, \quad \text{for all } x \in H,$$

or still

$$\langle x, y_1 - y_2\rangle = 0, \quad \text{for all } \ x \in H.$$

In particular for $x = y_1 - y_2$, we get

$$\langle y_1 - y_2, y_1 - y_2\rangle = \|y_1 - y_2\|^2 = 0.$$

Hence

$$y_1 = y_2$$

Theorem 2

Let A be an operator defined from a Hilbert space H_1 into a Hilbert space H_2 then, there exists an adjoint operator of A denoted by A^* defined from H_2 into H_1 such that

$$\langle A\varphi, \psi\rangle_{H_2} = \langle \varphi, A^*\psi\rangle_{H_1},$$

for all $\varphi \in H_1$ and $\psi \in H_2$. Besides, we have

$$\| A \| = \| A^* \| .$$

Proof

Define a linear functional U from H_1 into $K = (\mathbb{R} \text{ or } \mathbb{C})$, such that, for all $\varphi \in H_1$ and for all $\psi \in H_2$, we get

$$\begin{array}{ccc} H_1 & \to & K \\ \varphi & \mapsto & U(\varphi) = \langle A\varphi, \psi \rangle \end{array}$$

This functional is bounded, for

$$\begin{aligned} |U(\varphi)| &= |\langle A\varphi, \psi \rangle| \\ &\leq \|A\varphi\| \, \|\psi\| \\ &\leq \|A\| \, \|\varphi\| \, \|\psi\| \end{aligned}$$

By the Riesz representation theorem there exists a unique element $g \in H_1$ such that

$$U(\varphi) = \langle \varphi, g \rangle \,,$$

or still

$$U(\varphi) = \langle A\varphi, \psi \rangle = \langle \varphi, g \rangle \,,$$

this equality defines an adjoint operator of A denoted by A^* defined from H_2 into H_1 by

$$A^*\psi = g.$$

In other words, we write

$$\langle A\varphi, \psi \rangle = \langle \varphi, g \rangle = \langle \varphi, A^*\psi \rangle \,.$$

It is easy to verify that, the operator A^* is linear and unique, say

$$\begin{aligned} \langle \varphi, A^*(\alpha_1\psi_1 + \alpha_2\psi_2) \rangle &= \langle A\varphi, \alpha_1\psi_1 + \alpha_2\psi_2 \rangle \\ &= \overline{\alpha}_1 \langle A\varphi, \psi_1 \rangle + \overline{\alpha}_2 \langle A\varphi, \psi_2 \rangle \\ &= \overline{\alpha}_1 \langle \varphi, A^*\psi_1 \rangle + \overline{\alpha}_2 \langle \varphi, A^*\psi_2 \rangle \\ &= \langle \varphi, \alpha_1 A^*\psi_1 \rangle + \langle \varphi, \alpha_2 A^*\psi_2 \rangle \\ &= \langle \varphi, \alpha_1 A^*\psi_1 + \alpha_2 A^*\psi_2 \rangle \,. \end{aligned}$$

Besides, we get

$$\begin{aligned} \|A^*\psi\|^2 &= \langle A^*\psi, A^*\psi \rangle \\ &= \langle AA^*\psi, \psi \rangle \\ &\leq \|AA^*\psi\| \, \|\psi\| \\ &\leq \|A\| \, \|A^*\psi\| \, \|\psi\| \,, \end{aligned}$$

after simplification, we obtain

$$\|A^*\psi\| \leq \|A\| \, \|\psi\|$$

Hence, we get the following inequality

$$\|A^*\| \leq \|A\| , \tag{1}$$

Inversely,

$$\begin{aligned} \|A\varphi\|^2 &= \langle A\varphi, A\varphi \rangle \\ &= \langle \varphi, A^* A\varphi \rangle \\ &\leq \|\varphi\| \, \|A^* A\varphi\| \\ &\leq \|\varphi\| \, \|A^*\| \, \|A\varphi\| , \end{aligned}$$

after simplification, we obtain

$$\|A\varphi\| \leq \|A^*\| \, \|\varphi\|$$

Hence, we get the following inequality

$$\|A\| \leq \|A^*\| . \tag{2}$$

The relations (1) and (2), give the equality of the norms

$$\|A\| = \|A^*\| .$$

Theorem 3

Let A be a compact operator defined from a Hilbert space H_1 into a Hilbert space H_2 then, the adjoint operator A^* defined from H_2 into H_1 is also a compact operator.

Proof

Let ψ_n be a bounded sequence of H_2, that is to say, there exists a constant $M > 0$, such that $\|\psi_n\| < M$, for all $n \in \mathbb{N}$. The operator A is compact from H_1 into H_2 and the operator adjoint A^* is bounded from H_2 into H_1, then the operator AA^* defined from H_2 into H_2 is compact as product of two operators one compact and another bounded. Hence, there exists a subsequence ψ_{n_k} such that the sequence $AA^*(\psi_{n_k})$ converges in H_2. Say

$$\begin{aligned}\left\|A^*\psi_{n_p}-A^*\psi_{n_q}\right\|^2 &= \left\|A^*(\psi_{n_p}-\psi_{n_q})\right\|^2\\ &= \left\langle A^*(\psi_{n_p}-\psi_{n_q}), A^*(\psi_{n_p}-\psi_{n_q})\right\rangle\\ &= \left\langle AA^*(\psi_{n_p}-\psi_{n_q}), \psi_{n_p}-\psi_{n_q}\right\rangle\\ &\leq \left\|AA^*\left(\psi_{n_p}-\psi_{n_q}\right)\right\|\left\|\psi_{n_p}-\psi_{n(q)}\right\|\\ &= \left\|AA^*\psi_{n_p}-AA^*\psi_{n_q}\right\|\left\|\psi_{n_p}-\psi_{n(q)}\right\|\\ &\leq 2M\varepsilon,\end{aligned}$$

what shows that, the sequence $A^*\psi_{n_k}$ is of Cauchy into a Hilbert space H_2. Hence, the convergence of this sequence $A^*\psi_{n_k}$ in H_2. In other words, the operator A^* is compact.

Theorem 4 *(First Fredholm theorem)*
Let A be a compact operator defined from a Hilbert space H_1 into a Hilbert space H_2 then, the subspaces $N(I-A)$ and $N(I-A^*)$ have the same finite dimensional.

Proof
The subspaces $N(I-A)$ and $N(I-A^*)$ are of finite dimensional, so long as the operators A and A^* are compact.

Let $\varphi_1, \varphi_2, ..., \varphi_m$ be a basis of $N(I-A)$ of m dimension and $\psi_1, \psi_2, ...\psi_n$ a basis of $N(I-A^*)$ of n dimension, then there exists a two biorthogonal basis of linear functional $\varphi_1^*, \varphi_2^*, ..., \varphi_m^*$ and $\psi_1^*, \psi_2^*, ...\psi_n^*$ such that

$$\left\langle \varphi_i^*, \varphi_j\right\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j\end{cases}, \quad i,\ j=1,2,...,m\ ,$$

and

$$\left\langle \psi_i^*, \psi_j\right\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j\end{cases}, \quad i,\ j=1,2,...,n\ .$$

1. *First case* $m<n$

Let B be an operator defined on H_1 given by

$$B=A+V,$$

where V is an operator of finite dimensional defined as follows

$$V(\varphi) = \sum_{i=1}^{m} \langle \varphi, \varphi_i^* \rangle \psi_i^*. \tag{3}$$

Noting that, the operator V is compact because it applies H_1 in a space of finite dimensional. So the operator $B = A + V$ is compact operator as sum of two compact operators A and V.

Let φ_0 be an element of $N(I - B)$, then we write the equation

$$\begin{aligned} &(I - B)\,\varphi_0 = 0 \\ &\varphi_0 - A\varphi_0 - V\varphi_0 = 0, \end{aligned}$$

or explicitly

$$\varphi_0 - A\varphi_0 - \sum_{i=1}^{m} \langle \varphi_0, \varphi_i^* \rangle \psi_i^* = 0. \tag{4}$$

Let's multiply both sides by $\psi_j \in N(I - A^*)$, $j = 1, 2, 3, ..., m$, we get

$$\begin{aligned} &\left\langle \left(\varphi_0 - A\varphi_0 - \sum_{i=1}^{m} \langle \varphi_0, \varphi_i^* \rangle \psi_i^* \right), \psi_j \right\rangle = 0 \;\Rightarrow \\ &\left\langle \varphi_0 - A\varphi_0, \psi_j \right\rangle - \sum_{i=1}^{m} \langle \varphi_0, \varphi_i^* \rangle \left\langle \psi_i^*, \psi_j \right\rangle = 0 \quad \Rightarrow \\ &\left\langle \varphi_0 - A\varphi_0, \psi_j \right\rangle - \left\langle \varphi_0, \varphi_j^* \right\rangle \left\langle \psi_j^*, \psi_j \right\rangle = 0 \quad \Rightarrow \\ &\left\langle \varphi_0 - A\varphi_0, \psi_j \right\rangle - \left\langle \varphi_0, \varphi_j^* \right\rangle = 0 \quad \Rightarrow \\ &\left\langle \varphi_0 - A\varphi_0, \psi_j \right\rangle = \left\langle \varphi_0, \varphi_j^* \right\rangle. \end{aligned}$$

Noting that, $\psi_j \in N(I - A^*)$, this implies $(I - A^*)\psi_j = 0$ for all $j = 1, 2, ..., m$, then

$$\begin{aligned} \left\langle \varphi_0 - A\varphi_0, \psi_j \right\rangle &= \left\langle (I - A)\varphi_0, \psi_j \right\rangle \\ &= \left\langle \varphi_0, (I - A)^*\psi_j \right\rangle = 0 \end{aligned}$$

Hence, for all $j = 1, 2, ..., m$, we have

$$\left\langle \varphi_0, \varphi_j^* \right\rangle = 0. \tag{5}$$

The relations (4) and (5) give

$$\varphi_0 - A\varphi_0 = (I - A)\varphi_0 = 0. \tag{6}$$

In other words, we obtain $\varphi_0 \in N(I-A)$. Whence

$$\varphi_0 = \sum_{i=1}^{m} \alpha_i \varphi_i = \sum_{i=1}^{m} \langle \varphi_0, \varphi_i^* \rangle \varphi_i.$$

As one has $\alpha_i = \langle \varphi_0, \varphi_i^* \rangle$ and according to (5), we obtain $\varphi_0 = 0$.

All elements φ_0 of the subspace $N(I-B)$ are nulls. Hence the operator $(I-B)$ is injective and therefore the nonhomogeneous equation

$$\varphi - B\varphi = f,$$

admits a unique solution for all second member f, in particular for $f = \psi_{m+1}^*$. Say

$$(I-B)\varphi = \varphi - A\varphi - \sum_{i=1}^{m} \langle \varphi, \varphi_i^* \rangle \psi_i^* = \psi_{m+1}^*.$$

Suppose that this equation admits ξ as solution, then it comes

$$\begin{aligned} &(I-B)\,\xi = \psi_{m+1}^* \;\Rightarrow \\ &\xi - A\xi - \sum_{i=1}^{m} \langle \xi, \varphi_i^* \rangle \psi_i^* = \psi_{m+1}^* \end{aligned}$$

Let's multiply both sides of this equation by ψ_{m+1}, in order to obtain

$$\begin{aligned} &\left\langle \xi - A\xi - \sum_{i=1}^{m} \langle \xi, \varphi_i^* \rangle \psi_i^*, \psi_{m+1} \right\rangle = \left\langle \psi_{m+1}^*, \psi_{m+1} \right\rangle \\ &\left\langle \xi - A\xi, \psi_{m+1} \right\rangle - \sum_{i=1}^{m} \langle \xi, \varphi_i^* \rangle \left\langle \psi_i^*, \psi_{m+1} \right\rangle = 1 \\ &\left\langle \xi, (I-A^*)\psi_{m+1} \right\rangle - \sum_{i=1}^{m} \langle \xi, \varphi_i^* \rangle \left\langle \psi_i^*, \psi_{m+1} \right\rangle = 1. \end{aligned} \tag{7}$$

On the other hand, we have

$$\left\langle \xi, (I-A^*)\psi_{m+1} \right\rangle - \sum_{i=1}^{m} \langle \xi, \varphi_i^* \rangle \left\langle \psi_i^*, \psi_{m+1} \right\rangle = 0,$$

because the inner product $\left\langle \psi_i^*, \psi_{m+1} \right\rangle = 0$ for all $i = 1, 2, ...m$ and one has $\psi_{m+1} \in N(I-A^*)$, say $(I-A^*)\psi_{m+1} = 0$ and consequently $\left\langle \xi, (I-A^*)\psi_{m+1} \right\rangle$. Contradiction with (7) .Hence, we must have $m < n$ is not true.

2. *Second case* $m > n$

Let B^* be an operator defined on H_2 given by

$$B^* = A^* + V^*,$$

where V^* is an operator of finite dimensional defined as follows

$$V^*(\psi) = \sum_{k=1}^{n} \langle \psi, \psi_k^* \rangle \varphi_k^*. \tag{8}$$

Interchanging the roles, let ψ_0 be an element of $N(I - B^*)$ then, we write the equation

$$\begin{aligned} &(I - B^*)\,\psi_0 = 0 \\ &\psi_0 - A^*\psi_0 - V^*\psi_0 = 0, \end{aligned}$$

or explicitly,

$$\psi_0 - A^*\psi_0 - \sum_{k=1}^{n} \langle \psi_0, \psi_k^* \rangle \varphi_k^* = 0,$$

in order to obtain $m < n$ is not true, and thereafter, one leads to the equality $m = n$.

Theorem 5 *(Second Fredholm theorem)*
Let A be a compact operator defined from a Hilbert space H_1 into a Hilbert space H_2 then, the subspaces $R(I - A)$ and $R(I - A^*)$ have the following relation

$$\begin{aligned} N(I - A^*) &= R^{\perp}(I - A) \\ &\Leftrightarrow \\ N^{\perp}(I - A^*) &= R(I - A) \end{aligned}$$

end

$$\begin{aligned} N(I - A) &= R^{\perp}(I - A^*) \\ &\Leftrightarrow \\ N^{\perp}(I - A) &= R(I - A^*) \end{aligned}$$

Proof

1. *First case*

Suppose that, $\psi \in N(I-A^*)$, say $(I-A^*)\psi = 0$ then, for all φ in H_1, we have

$$\begin{aligned} & \langle \varphi, (I-A^*)\psi \rangle = 0 \\ \Rightarrow \quad & \langle (I-A)\varphi, \psi \rangle = 0, \end{aligned}$$

it follows that

$$\psi \in R^{\perp}(I-A),$$

or still

$$N(I-A^*) \subset R^{\perp}(I-A).$$

Inversely, suppose that, $\psi \in R^{\perp}(I-A)$ then, for all φ in H_1, we have

$$\begin{aligned} & \langle \psi, (I-A)\varphi \rangle = 0 \\ \Rightarrow \quad & \langle (I-A^*)\psi, \varphi \rangle = 0, \quad \forall \varphi \in H_1, \end{aligned}$$

it follows that

$$(I-A^*)\psi \in H_1^{\perp} = \{0\} \Leftrightarrow \psi \in N(I-A^*),$$

or still

$$R^{\perp}(I-A) \subset N(I-A^*).$$

Whence

$$R^{\perp}(I-A) = N(I-A^*) \quad \Leftrightarrow \quad R(I-A) = N^{\perp}(I-A^*)$$

2. *Second case*

Suppose that, $\varphi \in N(I-A)$, say $(I-A)\varphi = 0$ then, for all ψ in H_2, we have

$$\begin{aligned} & \langle (I-A)\varphi, \psi \rangle = 0 \\ \Rightarrow \quad & \langle \varphi, (I-A^*)\psi \rangle = 0, \quad \forall \psi \in H_2 \end{aligned}$$

it follows that

$$\varphi \in R^{\perp}(I-A^*),$$

or still

$$N(I-A) \subset R^{\perp}(I-A^*).$$

Inversely, suppose that, $\varphi \in R^{\perp}(I - A^*)$ then, for all ψ in H_2, we have

$$\langle \varphi, (I - A^*)\psi \rangle = 0$$
$$\Rightarrow \quad \langle (I - A)\varphi, \psi \rangle = 0, \quad \forall \psi \in H_2,$$

it follows that

$$(I - A)\varphi \in H_2^{\perp} = \{0\} \Leftrightarrow \varphi \in N(I - A),$$

or still

$$R^{\perp}(I - A^*) \subset N(I - A).$$

Whence

$$R^{\perp}(I - A^*) = N(I - A) \quad \Leftrightarrow \quad R(I - A^*) = N^{\perp}(I - A).$$

Theorem 6 *(Fredholm Alternative)*

Let A be a compact operator defined from a Hilbert space H into itself, then the equations

$$\varphi - A\varphi = f\ , \tag{9}$$

and its adjoint

$$\psi - A^*\psi = g, \tag{10}$$

admit a unique solutions for all second member if the homogeneous equations

$$\varphi - A\varphi = 0,$$

and

$$\psi - A^*\psi = 0,$$

admit uniquely the trivial solutions $\varphi = 0$ and $\psi = 0$.

Contrarily, if the homogeneous equations have the same number of solutions $\varphi_1, \varphi_2, ..., \varphi_n$ and $\psi_1, \psi_2, ..., \psi_n$ respectively then, the nonhomogeneous equations (9) and (10) have solutions if and only if, for all $k = 1, 2, ..., n$, we have

$$\langle f, \psi_k \rangle = 0,$$

and

$$\langle g, \varphi_k \rangle = 0.$$

The general solution of the equation (9) is given by

$$\varphi = \varphi_0 + \sum_{k=1}^{n} \alpha_k \varphi_k,$$

and the one of the equation (10) *is given by*

$$\psi = \psi_0 + \sum_{k=1}^{n} \alpha_k \psi_k,$$

where φ_0 *and* ψ_0 *are any particular solutions of the equations* (9) *and* (10) *respectively,* $a_1, a_2, ..., a_n$ *are arbitrary constants.*

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