§3. Equations with Compact Operators

Equations of the second kind

Let A be a compact operator defined on the normed space E into itself, the operator $T = I - A$ where I denotes the identity operator defines an operator equation called equation of the second kind, given as

$$T\varphi = (I - A)\varphi = f,$$

or merely

$$\varphi - A\varphi = f,$$

where f is a given function of E and φ is the unknown function of E.

Theorem 1

The null-space $N(T)$ of the operator T defined by

$$N(T) = \ker T = \{\varphi \in E; \;\; T\varphi = (I - A)\varphi = 0\},$$

is a closed and finite dimensional subspace.

Proof

Indeed, it is known that the kernel $N(T)$ of a bounded operator T is a closed subspace of E, since, for all sequence φ_n in $N(T)$ converges to φ in E, we obtain φ in $N(T)$. Really, due to the boundedness of T we have

$$T\varphi_n = 0 \;\Rightarrow\; \lim_{n\to\infty} T\varphi_n = 0,$$

or still

$$T\left(\lim_{n\to\infty} \varphi_n\right) = 0 \Rightarrow T(\varphi) = 0.$$

Hence, the null-space $N(T)$ is closed.

On the other hand, all functions $\varphi \in N(T)$ must satisfy the equation

$$T\varphi = \varphi - A\varphi = 0,$$

or still

$$A\varphi = \varphi.$$

Noting that, the restriction of the operator A to the subspace $N(T)$ coincides with the identity operator on $N(T)$. The operator A is compact from E to E and therefore also compact from $N(T)$ to $N(T)$ since $N(T)$ is closed. Evidently, for all bounded sequence φ_n in E in particular in $N(T)$ the sequence $A\varphi_n = \varphi_n$ contains a convergent subsequence $A\varphi_{n_k} = \varphi_{n_k}$ in the closed $N(T)$. Hence, the compact operator A represents the identity operator on the subspace $N(T)$ and therefore the subspace $N(T)$ is of finite dimensional.

Remark 1

The null-space $N(T^n)$ of the operator T^n for all $n \in \mathbb{N}$, is a closed and finite dimensional subspace. Indeed, the operators T^n can be written in the form

$$T^n = (I - A)^n = I - A_n,$$

where A_n is a compact operator as combination of compact operators given by

$$A_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} A^i$$

Theorem 2

The sequence of null-spaces sets

$$N(T),\ N(T^2), ...,\ N(T^n), ...$$

is increasing and stationary sequence. In other words, the sequence contains uniquely a finite number of distinct sets, so there exists a nonnegative integer $p \in N$, *such that*

$$\{0\} \subset N(T) \subset N\left(T^2\right) \subset ... \subset N\left(T^p\right) = N\left(T^{p+1}\right) = ...,$$

the number p is called the Riesz number of the compact operator A for the null-spaces sets $N(T^n)$.

Proof

Indeed, the inclusion is evident, since

$$\varphi \in N(T^n) \Rightarrow T^n\varphi = 0,$$

and therefore

$$T(T^n\varphi) = T^{n+1}\varphi = 0 \ \Rightarrow \ \varphi \in N(T^{n+1}).$$

Hence, the inclusion of sets

$$N(T^n) \subset N(T^{n+1}), \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Suppose that there is no integer $p \in \mathbb{N}$, such that the sequence $N(T^n)$ is stationary, that is to say

$$N(T^m) \neq N(T^n), \text{ for all } m, n \in \mathbb{N}, \text{ with } m < n.$$

In other words, we write

$$\{0\} \subset N(T) \subset ... \subset N(T^m) \subset N(T^{m+1}) \subset ... \subset N(T^{n-1}) \subset N(T^n) \subset ...$$

In particular, taking $N(T^{n-1}) \neq N(T^n)$, the relation $N(T^{n-1}) \subset N(T^n)$ between a closed subspaces involves the existence of an element φ_n in $N(T^n)$, with unit norm $\|\varphi_n\| = 1$, such that

$$\left\|\varphi_n - \varphi_{n-1}\right\| > \frac{1}{2}, \text{ for all } \ \varphi_{n-1} \in N(T^{n-1}).$$

Generally, for all sequence $\varphi_n \in N(T^n)$ and for all m, n such that $m < n$, we have the following relation

$$\begin{aligned} \|A\varphi_n - A\varphi_m\| &= \|(I-T)\varphi_n - (I-T)\varphi_m\| \\ &= \|\varphi_n - T\varphi_n - \varphi_m + T\varphi_m\| \\ &= \|\varphi_n - (\varphi_m - T\varphi_m + T\varphi_n)\| > \frac{1}{2}. \end{aligned} \tag{2}$$

For, the elements of the sequence $(\varphi_m - T\varphi_m + T\varphi_n)$ belong to the subspace $N(T^{n-1})$. Indeed, due to the relation

$$N(T^m) \subset N(T^{n-1}) \subset N(T^n),$$

it comes

$$\varphi_m \in N(T^m) \Rightarrow \varphi_m \in N(T^{n-1}) \ \text{ and } \varphi_m \in N(T^n),$$

or still

$$\varphi_m \in N(T^m) \Rightarrow T^m\varphi_m = 0, \ T^{n-1}\varphi_m = 0 \ \text{ and } \ T^n\varphi_m = 0.$$

Noting that, for $\varphi_n \in N(T^n)$, we get

$$T^{n-1}(\varphi_m - T\varphi_m + T\varphi_n) = T^{n-1}\varphi_m - T^n\varphi_m + T^n\varphi_n = 0.$$

Hence

$$(\varphi_m - T\varphi_m + T\varphi_n) \in N(T^{n-1}).$$

The sequence φ_n is bounded, so by virtue of the compactness of the operator A, we can extract a convergent subsequence from the sequence $A\varphi_n$. Contradiction with the relation (2). Hence

$$N(T^{n-1}) = N(T^n).$$

It remains to demonstrate now the relation

$$N(T^n) = N(T^{n+1}).$$

Indeed, for $\varphi \in N(T^{n+1})$ we get

$$\varphi \in N(T^{n+1}) \Rightarrow T^{n+1}\varphi = T^n(T\varphi) = 0,$$

it gives

$$T\varphi \in N(T^n) = N(T^{n-1}),$$

that means

$$T\varphi \in N(T^{n-1}) \Rightarrow T^{n-1}(T\varphi) = T^n\varphi = 0 \Rightarrow \varphi \in N(T^n),$$

and therefore

$$N(T^{n+1}) \subset N(T^n).$$

Hence, there exists a nonnegative integer $p \in \mathbb{N}$, such that

$$\{0\} \subset N(T) \subset N(T^2) \subset ... \subset N(T^p) = N(T^{p+1}) = N(T^{p+2}) = ...,$$

where p is given by

$$p = \min\{k \in \mathbb{N};\ \text{such that } N(T^k) = N(T^{k+1})\}.$$

Theorem 3
The range space $R(T)$ of the operator T defined by

$$R(T) = \operatorname{Im} T = T(E) = \{\psi;\ \ \exists\varphi \in E,\ T\varphi = \psi\},$$

is a closed subspace.

Proof
It is known that, the range $R(T)$ of a linear operator T is a linear subspace. Let f be an element of the closure $\overline{T(E)}$, then there exists a sequence f_n of the set $T(E)$ such that

$$\lim_{n\to\infty} f_n = f.$$

In other words, $f_n \in T(E)$ there exists a sequence $\varphi_n \in E$ such that

$$T\varphi_n = f_n,$$

with the relation of convergence

$$\lim_{n\to\infty} T\varphi_n = \lim_{n\to\infty} f_n = f.$$

- *First case* φ_n *bounded*

Suppose that, the sequence φ_n is bounded then, due to the compactness of the operator A there exists a subsequence $A\varphi_{n(k)}$ from the sequence $A\varphi_n$ such that, $A\varphi_{n(k)}$ converges to ψ. Hence, the convergence of the subsequence $\varphi_{n(k)}$ to an element φ in E. Say

$$\begin{aligned}
\lim_{k\to\infty} \varphi_{n(k)} &= \lim_{k\to\infty} \left(T\varphi_{n(k)} + A\varphi_{n(k)}\right) \\
&= \lim_{k\to\infty} T\varphi_{n(k)} + \lim_{k\to\infty} A\varphi_{n(k)} \\
&= f + \psi = \varphi \in E.
\end{aligned}$$

Due to the boundedness of the operator T and the convergence of the sequence $T\varphi_n$, we get

$$\begin{aligned}
f = \lim_{n\to\infty} f_n &= \lim_{n\to\infty} T\varphi_n \\
&= \lim_{k\to\infty} T\varphi_{n(k)} = T\left(\lim_{k\to\infty} \varphi_{n(k)}\right) = T\varphi.
\end{aligned}$$

Hence $f = T\varphi \in T(E) = \overline{T(E)}$.

- *Second case* φ_n *unbounded*

Suppose that, the sequence φ_n is not bounded, then we get

1. If $\varphi_n \in N(T)$

For the sequence φ_n inthe null space $N(T)$, we have $T\varphi_n = f_n = 0$

$$T\varphi_n = 0 \Rightarrow \lim_{n\to\infty} T\varphi_n = 0 \Rightarrow f = 0 \in T(E) = \overline{T(E)},$$

as a linear subspace contains the null element.

2. If $\varphi_n \notin N(T)$

Taking the subspace G of E spanned by φ_n and $N(T)$ defined as

$$G = \text{span } \{\varphi_n + N(T)\}.$$

The subspace $N(T)$ is closed in G. Hence, there exists an element $\psi_n \in G$ with a unit norm $\|\psi_n\| = 1$ such that

$$\|\psi_n - \xi_n\| > \frac{1}{2}, \quad \forall \xi_n \in N(T),$$

with the following relation

$$\psi_n = a_n\varphi_n + \theta_n \ , \quad a_n \in \mathbb{R}, \quad \theta_n \in N(T).$$

Noting that, there is no subsequence $a_{n(k)}$ of the sequence a_n converges to the null element. For, if we suppose there exists a such subsequence, say $\lim_{k\to\infty} a_{n(k)} = 0$, we obtain

$$\begin{aligned}
\lim_{k\to\infty} T\psi_{n(k)} &= \lim_{k\to\infty} (a_{n(k)}T\varphi_{n(k)}) + \lim_{k\to\infty} T\theta_{n(k)} \\
&= \lim_{k\to\infty} a_{n(k)} . \lim_{k\to\infty} T\varphi_{n(k)} + \lim_{k\to\infty} T\theta_{n(k)} \\
&= 0f + 0 = 0.
\end{aligned}$$

In other words, there exists a subsequence $\psi_{n(j)}$ of the subsequence $\psi_{n(k)}$ of the bounded sequence ψ_n such that $A\psi_{n(j)}$ converges to an element ψ of E. This implies the convergence of the subsequence $\psi_{n(j)}$ to the same element ψ of E, for, we have

$$\lim_{j\to\infty} \psi_{n(j)} = \lim_{j\to\infty} T\psi_{n(j)} + \lim_{j\to\infty} A\psi_{n(j)} = \psi.$$

It is clear that, $T\psi = 0$. Hence $\psi \in N(T)$. Contradiction with the fact that

$$\|\psi_n - \xi_n\| > \frac{1}{2}, \quad \forall \xi_n \in N(T).$$

We can therefore conclude that a_n^{-1} is bounded. Say

$$a_n^{-1}\psi_n = \varphi_n + a_n^{-1}\theta_n.$$

Then, it comes

$$\begin{aligned}
\lim_{n\to\infty} T(a_n^{-1}\psi_n) &= \lim_{n\to\infty} T\varphi_n + \lim_{n\to\infty} T(a_n^{-1}\theta_n) \\
&= \lim_{n\to\infty} T\varphi_n + 0 = f.
\end{aligned}$$

The sequence $a_n^{-1}\psi_n$ is bounded as product of two bounded sequences a_n^{-1} and ψ_n. Hence there exists a subsequence $a_{n(k)}^{-1}\psi_{n(k)}$ such that $A(a_{n(k)}^{-1}\psi_{n(k)})$ converges to an element $a^{-1}\psi$ of E. This implies the convergence of the subsequence $a_{n(k)}^{-1}\psi_{n(k)}$ to the same element $a^{-1}\psi$ of E, for, we have

$$\lim_{k\to\infty} a_{n(k)}^{-1}\psi_{n(k)} = \lim_{k\to\infty} T(a_{n(k)}^{-1}\psi_{n(k)}) + \lim_{k\to\infty} A(a_{n(k)}^{-1}\psi_{n(k)}) = a^{-1}\psi \in E.$$

The operator T is continuous, then we write

$$\begin{aligned} \lim_{k\to\infty} T(a_{n(k)}^{-1}\psi_{n(k)}) &= T(\lim_{k\to\infty}(a_{n(k)}^{-1}\psi_{n(k)})) \\ &= T(a^{-1}\psi) = f \in T(E) = \overline{T(E)}. \end{aligned}$$

Hence, the result

$$T(E) = \overline{T(E)}.$$

Theorem 4

The sequence of range spaces sets

$$R(T), R(T^2), ..., \; R(T^n), ...$$

is decreasing and stationary sequence. In other words, the sequence contains uniquely a finite number of distinct sets, so there exists a nonnegative integer $q \in N$, *such that*

$$..... = R(T^{q+1}) = R(T^q) \subset ...,. \subset R(T^2) \subset R(T) \subset E$$

The number q is called the Riesz number of the compact operator A for the range spaces sets $R(T^n)$.

Proof

Indeed, the inclusion is evident, since

$$\psi \in R(T^{n+1}) \Rightarrow \psi = T^{n+1}(\varphi) = T^n(T\varphi) = T^n\varphi_1 \in R(T^n),$$

and therefore

$$\psi = T^{n+1}\varphi \quad \Rightarrow \quad \psi = T^n\varphi_1.$$

Hence, the inclusion of sets

$$R(T^{n+1}) \subset R(T^n). \tag{3}$$

Suppose that there is no integer $q \in \mathbb{N}$, such that the sequence $R(T^{\;n})$ is stationary, that is to say

$$R(T^m) \neq R(T^n), \text{ for all } m, n \in \mathbb{N}, \text{ with } n < m.$$

In other words, we have

$$... \subset R(T^m) ... \subset R(T^{n+1}) \subset R(T^n) \subset ... \subset R(T) \subset E$$

In particular, taking $R(T^{n+1}) \neq R(T^n)$, the relation $R(T^{n+1}) \subset R(T^n)$ between a closed subspaces involves the existence of an element ψ_n in $R(T^n)$, with unit norm $\|\psi_n\| = 1$, such that

$$\| \psi_n - \psi_{n+1} \| > \frac{1}{2}, \text{ for all } \psi_{n+1} \in R(T^{n+1}).$$

Generally, for all sequence $\psi_n \in R(T^n)$ and for all m, n such that $n < m$, we have the following relation

$$\begin{aligned} \|A\psi_n - A\psi_m\| &= \|(I - T)\psi_n - (I - T)\psi_m\| \\ &= \|\psi_n - T\psi_n - \psi_m + T\psi_m\| \\ &= \|\psi_n - (\psi_m - T\psi_m + T\psi_n)\| > \frac{1}{2}. \end{aligned} \tag{4}$$

For, the elements of the sequence $(\psi_m - T\psi_m + T\psi_n)$ belong to the subspace $N(T^{n+1})$. Indeed, due to the relation

$$R(T^{m+1}) \subset R(T^m) \subset R(T^{n+1}) \subset R(T^n),$$

it comes

$$\psi_m \in R(T^m) \Rightarrow \psi_m \in R(T^{n+1}),$$

also, we have

$$T\psi_m \in R(T^{m+1}) \Rightarrow T\psi_m \in R(T^{n+1}).$$

Noting that, for $\psi_n \in R(T^n)$, we get

$$\psi_n \in R(T^n) \Rightarrow T\psi_n \in R(T^{n+1})$$

Hence

$$(\psi_m - T\psi_m + T\psi_n) \in R(T^{n+1}).$$

The sequence ψ_n is bounded, so by virtue of the compactness of the operator A, we can extract a convergent subsequence from the sequence $A\psi_n$. Contradiction with the relation (4). Hence

$$R(T^{n+1}) = R(T^n).$$

It remains to demonstrate now the relation

$$R(T^{n+2}) = R(T^{n+1}).$$

Indeed, the first inclusion $R(T^{n+2}) \subset R(T^{n+1})$ is always true following (3), for the second one, we get

$$\begin{aligned} \psi \in R(T^{n+1}) \Rightarrow \psi &= T^{n+1}\varphi = T(T^n\varphi) \\ &= T(T^{n+1}\varphi_1) = T^{n+2}\varphi_1 \in R(T^{n+2}), \end{aligned}$$

or still

$$R(T^{n+1}) \subset R(T^{n+2}).$$

Hence, there exists a nonnegative integer $q \in \mathbb{N}$, such that

$$.... = R(T^{q+2}) = R(T^{q+1}) = R(T^q) \subset ...R(T^2) \subset R(T) \subset E$$

where q is given by

$$q = \min\{k \in \mathbb{N};\ \text{such that } R(T^k) = R(T^{k+1})\}.$$

Lemma 1

The Riesz number p of the null-spaces sets $N(T^n)$ and the Riesz number q of the ranges spaces $R(T^n)$ are equal. Say

$$p = q$$

Proof

Suppose that, the Riesz numbers p and q are different, say $p \neq q$.

1. *First case* $p < q$,

$$\{0\} \subset N(T) \subset ... \subset N(T^p) = N(T^{p+1}) = ... = N(T^{q-1}) = N(T^q) = .., \quad (5)$$

and also

$$\ldots = R(T^{q+1}) = R(T^q) \subset R(T^{q-1}) \subset \ldots \subset R(T^p) \subset \ldots \subset R(T) \subset E. \quad (6)$$

We can see that, there exists a function $\psi \in R(T^{q-1})$ such that $\psi \notin R(T^q)$, that is to say

$$\psi = T^{q-1}\varphi \in R(T^{q-1}),$$

the composition by the operator T of both sides, gives us

$$T\psi = T^q\varphi \in R(T^q) = R(T^{q+1}),$$

this relation shows that, there exists a function φ_1 such that

$$T\psi = T^q\varphi = T^{q+1}\varphi_1,$$

or still

$$T^{q+1}\varphi_1 - T^q\varphi = 0.$$

Hence, we obtain

$$T^q(T\varphi_1 - \varphi) = 0 \Rightarrow T\varphi_1 - \varphi \in N(T^q) = N(T^{q-1}). \quad (7)$$

It is to remark that the relation (7) gives us

$$T(\varphi_1) - \varphi \in N(T^{q-1}) \Rightarrow T^{q-1}(T\varphi_1 - \varphi) = 0 \Leftrightarrow T^q\varphi_1 = T^{q-1}\varphi = \psi,$$

this implies that $\psi = T^q\varphi_1 \in R(T^q)$, contradiction with the fact that $\psi \notin R(T^q)$.

2. *Second case* $q < p$,

$$\{0\} \subset N(T) \subset \ldots \subset N(T^q) \subset \ldots \subset N(T^{p-1}) \subset N(T^p) = N(T^{p+1}) = \ldots \quad (8)$$

and also

$$\ldots = R(T^p) = R(T^{p-1}) = \ldots = R(T^q) \subset R(T^{q-1}) \subset \ldots \subset R(T) \subset E \quad (9)$$

We can see that, there exists a function $\varphi \in N(T^p)$ such that $\varphi \notin N(T^{p-1})$, that is to say

$$T^{p-1}\varphi \in R(T^{p-1}) = R(T^p) = R(T^q),$$

this relation shows that, there exists functions φ_1 and φ_2 such that

$$T^{P-1}\varphi = T^p\varphi_1 = T^q\varphi_2, \tag{10}$$

the composition by the operator T of both sides and the relation $\varphi \in N(T^p)$ give us

$$T^p\varphi = T^{p+1}\varphi_1 = T^{q+1}\varphi_2 = 0,$$

this implies that

$$\varphi_1 \in N(T^{p+1}) = N(T^p).$$

Hence, it comes

$$T^{p+1}\varphi_1 = T^p\varphi_1 = 0.$$

It is to remark that the relation (10) gives us $T^p\varphi_1 = T^{p-1}\varphi = 0$, this implies that $\varphi \in N(T^{p-1})$, contradiction with the fact that $\varphi \notin N(T^{p-1})$.

Theorem 5

The subspaces $N(T^r)$ and $R(T^r)$ are supplementary. That is to say

$$E = \ker T^r \oplus \operatorname{Im} T^r = N(T^r) \oplus R(T^r),$$

where $r = p = q$ is the Riesz number.

Proof

For all element $\psi \in E$, we get

$$\psi \in E \Rightarrow T^r\psi \in R(T^r) = ... = R(T^{2r}).$$

This relation implies the existence of a function φ, such that

$$T^r\psi = T^{2r}\varphi \Rightarrow T^r(\psi - T^r\varphi) = 0,$$

or still,

$$(\psi - T^r\varphi) = \theta \in N(T^r).$$

Therefore, we have

$$\psi = \theta + T^r\varphi, \quad \text{with} \quad \theta \in N(T^r) \ \text{ and } T^r\varphi \in R(T^r).$$

For all element $\psi \in N(T^r) \cap R(T^r)$, we get

$$\psi \in R(T^r) \text{ and } \psi \in N(T^r),$$

this relation implies $T^r\psi = 0$ and the existence of a function φ, such that $\psi = T^r\varphi$, it comes

$$\psi = T^r\varphi \quad \Rightarrow \ T^r\psi = 0 = T^{2r}\varphi,$$

or still,

$$\varphi \in N(T^{2r}) = ..., ... = N(T^r).$$

Therefore, we have

$$\psi = T^r\varphi = 0.$$

Lemma 2

The operator $T = I - A$ is injective if and only if, the operator T^r is injective for all $r \in \mathbb{N}$.

Proof

Supposing that, The operator T is injective then, for all $r \in \mathbb{N}$, we have

$$\begin{aligned} T^r\varphi_1 = T^r\varphi_2 \ &\Rightarrow \ T(T^{r-1}\varphi_1) = T(T^{r-1}\varphi_2) \Rightarrow T^{r-1}\varphi_1 = T^{r-1}\varphi_2 \\ &\Rightarrow \ T(T^{r-2}\varphi_1) = T(T^{r-2}\varphi_2) \Rightarrow T^{r-2}\varphi_1 = T^{r-2}\varphi_2 \\ &\Rightarrow \ ...T(T\varphi_1) = T(T\varphi_2) \Rightarrow T\varphi_1 = T\varphi_2 \\ &\Rightarrow \ \varphi_1 = \varphi_2. \end{aligned}$$

Hence, the operator T^r is injective.

Supposing that, The operator T^r is injective for all $r \in \mathbb{N}$, then we have

$$\begin{aligned} T\varphi_1 = T\varphi_2 \ &\Rightarrow \ T^{r-1}(T\varphi_1) = T^{r-1}(T\varphi_2) \Rightarrow T^r\varphi_1 = T^r\varphi_2 \\ &\Rightarrow \ \varphi_1 = \varphi_2. \end{aligned}$$

Hence, the operator T is injective. That is to say

$$\{0\} = N(T) = N(T^2) = ... = N(T^r) = ..., ...$$

Lemma 3

The operator $T = I - A$ is surjective if and only if, the operator T^r is surjective for all $r \in \mathbb{N}$.

Proof

Suppose that, the operator T is surjective then, for all $r \in \mathbb{N}$, say

$$\begin{aligned}\forall \psi \in E,\ \exists \varphi_1 \in E;\ \psi = T\varphi_1 \ &\Rightarrow\ \exists \varphi_2 \in E;\ \varphi_1 = T\varphi_2\\ &\Rightarrow\ \psi = T\varphi_1 = T(T\varphi_2) = T^2\varphi_2\\ &\Rightarrow\ ..\exists \varphi_r \in E;\ \varphi_{r-1} = T\varphi_r\\ &\Rightarrow\ \psi = T\varphi_1 = T(T\varphi_2) = ... = T(T^{r-1}\varphi_r) = T^r\varphi_r.\end{aligned}$$

Finally, we obtain

$$\forall \psi \in E,\ \exists \varphi_r \in E;\ \psi = T^r\varphi.$$

Hence, the operator T^r is surjective.

Suppose that, the operator T^r is surjective for all $r \in \mathbb{N}$, say

$$\forall \psi \in E,\ \exists \varphi_1 \in E;\ \psi = T^r\varphi_1,$$

we can also write

$$T^r\varphi_1 = T(T^{r-1}\varphi_1) = \ T\varphi,$$

where the functîon $\varphi = T^{r-1}\varphi_1 \in E$. Finally, we obtain

$$\forall \psi \in E,\ \exists \varphi = T^{r-1}\varphi_1 \in E;\ \psi = T\varphi.$$

Hence, the operator T is surjective. That is to say

$$E = R(T) = R(T^2) = ... = R(T^r) = ..., ...$$

Theorem 6

Let A be a compact operator defined from a Banach space E into itself then, the operator $T = I - A$ is injective if and only if, $T = I - A$ is surjective. Besides the inverse operator $T^{-1} = (I - A)^{-1}$ defined from E into E is bounded.

Proof

It is known that, for all Riesz number $r = p = q$, The subspaces $N(T^r)$ and $R(T^r)$ are supplementary. Say

$$E = N(T^r) \oplus R(T^r).$$

- The injection of the operator T implies the one of T^r. Hence the surjection of the operator T^r which it assures us the surjection of the operator T.

- The surjection of the operator T implies the one of T^r. Hence the injection of the operator T^r which it assures us the injection of the operator T.

- The injection of the operator T or its surjection implies the bijection of this operator $T = (I - A)$. Hence the boundedness of its inverse $T^{-1} = (I - A)^{-1}$.

Theorem 7

Let A be a compact operator from a Banach space E into itself then, the nonhomogeneous equation

$$T\varphi = \varphi - A\varphi = f \tag{11}$$

admits a unique solution $\varphi \in E$, for all $f \in E$, if and only if, the homogeneous equation

$$T\varphi = \varphi - A\varphi = 0 \tag{12}$$

admits uniquely a trivial solution $\varphi = 0$.

Proof

Indeed, suppose that the equation (11) admits a solution for all $f \in E$, it wants to say that the operator T is surjective and the Riesz number r is null. Hence the operator T is injective. In other words, the equation (12) admits the trivial solution $\varphi = 0$.

Reciprocally, suppose that the equation (12) admits uniquely the trivial solution $\varphi = 0$, it wants to say that the operator T is injective and the Riesz number r is null. Hence, the operator T is surjective and therefore this operator is bijective. In other words, the existence and the uniqueness of the solution of the equation (11).

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