Some results on the Nadir's operator $N = AB^* - BA^*$

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Abstract

In this paper, we present some new results for the operator $N = AB^* - BA^*$ and study the invertibility of $I - N$ in the algebra $B(H)$ of all bounded linear operators on a complex separable Hilbert space H.

1 Introduction

In mathematics, the commutator gives an indication of the extent to which a certain binary ....

The operator $N = AB^* - BA^*$ of two operators acting on a Hilbert space is a central concept in quantum mechanics, since it quantifies how well the two observables described by these operators can be measured simultaneously. From this definition, we can see that if the operator N is zero, then $AB^* = BA^*$ so, the order of which the two corresponding measurements are applied to the physical system does not matter. On the contrary, if it is non-zero, then the order does matter. Finally, This operaator will be called Nadir's operator.

1.1 Main results

Theorem 1 *Let $B(H)$ be a Banach algebra with unit element I, then for all operators A and B in $B(H)$. The operator $N = AB^* - BA^*$ is is a Skew self-adjoint operator and never equal to the indetity I. In other words*

$$N = AB^* - BA^* \neq I$$

Proof. Indeed, we have

$$\begin{aligned} N^* &= (AB^* - BA^*)^* \\ &= (AB^*)^* - (BA^*)^* \\ &= BA^* - AB^* \\ &= -(AB^* - BA^*) \\ &= -N. \end{aligned}$$

Assume that, $N = AB^* - BA^* = I$, it follows $N^* = (AB^* - BA^*)^* = I^* = I$. Hence, from the relation $N^* = -N$, we get

$$I = -I.$$

Contradiction ■

Proposition 1 *The operator $N = AB^* - BA^*$ is normal*

Indeed, it follows from the theorem 1

$$\begin{aligned} NN^* &= (AB^* - BA^*)(AB^* - BA^*)^* \\ &= N(-N) \\ &= (-N)N \\ &= N^*N \end{aligned}$$

Corollary 1 *The operator N^2 is negative, that is to say $\langle N^2x, x\rangle \leq 0$ for all non-zero vectors x in H.*

Indeed, it is known that the operator NN^* is always positive and from the proposition 1 $NN^* = (-N)N = -N^2$

Theorem 2 *Let A, B be a bounded operator on the Hilbert space H where one of the operators A and B is compact, then the operator $N = AB^* - BA^*$ is compact with the operator $I - N$ is invertible.*

- *First case A is compact B bounded*

A compact$\Rightarrow$ A^* compacr$\Rightarrow$ AB^* and BA^* are compacts. Hence N is compact.

- *Second case B is compact A bounded*

B compact$\Rightarrow$ B^* compacr$\Rightarrow$ AB^* and BA^* are compacts. Hence N is compact.

Besides, it is known that, the operator N is never equal to the identity I, then $N - I \neq 0$ and so, $N - I$ is injective. Hence $N - I$ is bijective.

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